1 (a)

$f(x,y) = 2x - 3y + 1$

problem

$y' = 2x - 3y + 1$

$y(1) = 5$

$x_0 = 1, y_0 = 5$

Euler

$x_n = x_{n-1} + h$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

These become:

$$x_n = x_{n-1} + 0.05$$
$$y_n = y_{n-1} + (0.05)(2x_{n-1} - 3y_{n-1} + 1)$$

$x_0 = 1$

$y_0 = 5$

given

$x_1 = x_0 + h = 1 + 0.05 = 1.05$

$y_1 = y_0 + h(2x_0 - 3y_0 + 1)$

$= 5 + (0.05)(2(1) - 3(5) + 1)$

$= 4.4$

$x_1 = 1.05$

$y_1 = 4.4$

$$x_2 = x_1 + h = 1.05 + 0.05 = 1.1$$
$$y_2 = y_1 + h(2x_1 - 3y_1 + 1)$$
$$= 4.4 + (0.05)(2(1.05) - 3(4.4) + 1)$$
$$= 3.895$$

$x_2 = 1.1$
$y_2 =$
3.895

$$x_3 = x_2 + h = 1.1 + 0.05 = 1.15$$
$$y_3 = y_2 + h(2x_2 - 3y_2 + 1)$$
$$= 3.895 + (0.05)(2(1.1) - 3(3.895) + 1)$$
$$= 3.47075$$

$x_3 = 1.15$
$y_3 =$
3.47075

$$x_4 = x_3 + h = 1.15 + 0.05 = 1.2$$
$$y_4 = y_3 + h(2x_3 - 3y_3 + 1)$$
$$= 3.47075 + (0.05)(2(1.15) - 3(3.47075) + 1)$$
$$= 3.11514$$

$x_4 = 1.2$
$y_4 = 3.11514$

$$x_5 = x_4 + h = 1.2 + 1.05$$
$$y_5 = y_4 + h(2x_4 - 3y_4 + 1)$$
$$= 3.11514 + (0.05)(2(1.2) - 3(3.11514) + 1)$$
$$= 2.81787$$

$x_5 = 1.05$
$y_5 = 2.81787$

So, the approximation to the solution at $x = 1.25$ is 2.81787

① (b)

Let's solve this linear first-order ODE:

$$y' = 2x - 3y + 1$$

$$y' + 3y = 2x + 1$$

$$e^{3x} y' + 3e^{3x} y = (2x+1)e^{3x}$$

$$\left(ye^{3x}\right)' = (2x+1)e^{3x}$$

$$ye^{3x} = \int 2xe^{3x}\,dx + \int e^{3x}\,dx$$

combine these

$$\int 2xe^{3x}\,dx = \frac{2}{3}xe^{3x} - \frac{2}{3}\int e^{3x}$$

$u = 2x$ $\quad du = 2dx$

$dv = e^{3x}dx$ $\quad v = \frac{1}{3}e^{3x}$

$$\int u\,dv = uv - \int v\,du$$

$$ye^{3x} = \frac{2}{3}xe^{3x} + \frac{1}{3}\int e^{3x}\,dx$$

$$ye^{3x} = \frac{2}{3}xe^{3x} + \frac{1}{9}e^{3x} + C$$

$$\boxed{y = \frac{2}{3}x + \frac{1}{9} + Ce^{-3x}}$$

$$5 = y(1) = \frac{2}{3}(1) + \frac{1}{9} + Ce^{-3(1)}$$

$$5 = \frac{7}{9} + Ce^{-3}$$

$$5 - \frac{7}{9} = Ce^{-3}$$

$$\frac{38}{9} = Ce^{-3}$$

$$C = \frac{38}{9}e^{3}$$

Thus, the actual solution is

$$y = \frac{2}{3}x + \frac{1}{9} + \left(\frac{38}{9}e^{3}\right)e^{-3x}$$

or

$$\boxed{y = \frac{2}{3}x + \frac{1}{9} + \frac{38}{9}e^{3-3x}}$$

We get

$$y(1.25) = \frac{2}{3}(1.25) + \frac{1}{9} + \frac{38}{9}e^{3-3(1.25)}$$

$$\approx \boxed{2.93888}$$

This compares to the approximation $\boxed{y_5 = 2.81787}$

They differ by $\boxed{0.12101}$

2

$f(x,y) = 1 + y^2$

problem

$y' = 1 + y^2$

$y(0) = 0$

Euler

$x_n = x_{n-1} + h$

$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$

$x_0 = 0, y_0 = 0$

These become:

$x_n = x_{n-1} + 0.1$

$y_n = y_{n-1} + (0.1)(1 + y_{n-1}^2)$

$x_0 = 0$

$y_0 = 0$

given

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = 0 + (0.1)(1 + 0^2) = 0.1$

$x_1 = 0.1$

$y_1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y_2 = y_1 + (0.1)(1 + y_1^2)$

$= 0.1 + (0.1)(1 + 0.1^2) = 0.201$

$x_2 = 0.2$

$y_2 = 0.201$

$$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$$

$$y_3 = y_2 + (0.1)(1 + y_2^2)$$

$$= 0.201 + (0.1)(1 + 0.201^2)$$

$$= 0.30504$$

$x_3 = 0.3$

$y_3 = 0.30504$

$$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$$

$$y_4 = y_3 + (0.1)(1 + y_3^2)$$

$$= 0.30504 + (0.1)(1 + 0.30504^2)$$

$$= 0.414345$$

$x_4 = 0.4$

$y_4 = 0.414345$

$$x_5 = x_4 + h = 0.4 + 0.1 = 0.5$$

$$y_5 = y_4 + h(1 + y_4^2)$$

$$= 0.414345 + (0.1)(1 + 0.414345^2)$$

$$= 0.531513$$

$x_5 = 0.5$

$y_5 = 0.531513$

We approximate that the solution at $x = 0.5$ is around 0.531513

3

$$f(x,y) = xy + \sqrt{y}$$

problem

$$y' = xy + \sqrt{y}$$
$$y(0) = 1$$

Euler

$$x_n = x_{n-1} + h$$
$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

$$x_0 = 0, \; y_0 = 1$$

These become:

$$x_n = x_{n-1} + 0.1$$
$$y_n = y_{n-1} + (0.1)(x_{n-1} y_{n-1} + \sqrt{y_{n-1}})$$

$$x_0 = 0$$
$$y_0 = 1$$

given

$$x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1$$
$$y_1 = y_0 + (0.1)(x_0 y_0 + \sqrt{y_0})$$
$$= 1 + (0.1)((0)(1) + \sqrt{1})$$
$$= 1.1$$

$$x_1 = 0.1$$
$$y_1 = 1.1$$

$$x_2 = x_1 + 0.1 = 0.1 + 0.1 = 0.2$$

$$y_2 = y_1 + (0.1)(x_1 y_1 + \sqrt{y_1})$$

$$= 1.1 + (0.1)((0.1)(1.1) + \sqrt{1.1})$$

$$\approx 1.21588$$

$x_2 = 0.2$

$y_2 = 1.21588$

$$x_3 = x_2 + 0.1 = 0.2 + 0.1 = 0.3$$

$$y_3 = y_2 + (0.1)(x_2 y_2 + \sqrt{y_2})$$

$$= 1.21588 + (0.1)((0.2)(1.21588 + \sqrt{1.21588})$$

$$= 1.36262$$

$x_3 = 0.3$

$y_3 = 1.36262$

$$x_4 = x_3 + 0.1 = 0.3 + 0.1 = 0.4$$

$$y_4 = y_3 + (0.1)(x_3 y_3 + \sqrt{y_3})$$

$$= 1.36262 + (0.1)((0.3)(1.36262) + \sqrt{1.36262})$$

$$= 1.53386$$

$x_4 = 0.4$

$y_4 = 1.53386$

$$x_5 = x_4 + 0.1 = 0.4 + 0.1 = 0.5$$

$$y_5 = y_4 + (0.1)(x_4 y_4 + \sqrt{y_4})$$

$$= 1.53386 + (0.1)((0.4)(1.53386) + \sqrt{1.53386})$$

$$= 1.7344$$

$x_5 = 0.5$

$y_5 = 1.7344$

We approximate the actual solution at $x = 0.5$ to be approximately 1.7344